Abstract: In this essay, we read McKenzie’s seventeen paragraphs of 1957 on “demand theory without a utility index” as an opening to the narrative of 20th-century demand theory, and as a lever for the understanding of what has now reached culmination as the neoclassical theory of demand. In tracking the influence of these paragraphs on both theoretical and applied work, we also use them as a foothold for reflection on the process of theorizing, to argue for the view that one cannot neglect the problematic that the theory is adduced to address, that the historical narrative behind a particular theorem is indispensable in understanding the theorem itself. This implicit theorizing of the process of theorizing then forces us to consider Stigler’s distinction between textual and scientific exegesis, and confront it to a second-order level of theorizing, and thereby bring out the continuity, possibly not quite seamless, between a theorem and its history. (151 words.)

Keywords: Minimum income function, money-metric, expenditure function, indirect utility function, duality, adjoint, non-transitivities, incompleteness, indivisibilities, consumer’s surplus, index numbers, complementarity, general equilibrium theory, second-order theory, inside-outside, exegesis, precursor.

JEL Classifications: B31, B21, C61, D11.
The simplest things are often the most complicated to understand fully. Samuelson 1974.\textsuperscript{1}

While the surface matters are simple and can be put briefly, their investigation uncovers elaborate features. Much time can be spent mastering these. Standard mathematics often take a unique turn. Afriat 1980.\textsuperscript{2}

... I should have thought that what I had wielded was, in any case, Occam’s razor, but I am prepared not to argue that point. McKenzie 1958.\textsuperscript{3}

... there was a new tone but also a speculative power, as neither could just receive something without recreating it. Minds too big to repeat. Levinas 1986.\textsuperscript{4}

\textsuperscript{1}The first paragraph of Samuelson’s essay on “complementarity”. And for the reader who wonders what the subsequent quote from Levinas is doing in this essay, let us, if only provisionally, say that Samuelson’s silent interlocutor throughout his essay is Ludwig Wittgenstein.

\textsuperscript{2}See Afriat (1980, p. 1).

\textsuperscript{3}The OED has the following entry for Occam’s razor: “Attributed to Occam but of earlier origin, this principle is often quoted in Latin form \textit{Entia non sunt multiplicanda praeter necessitatem}. It is not found in this form in Occam’s writings, although he frequently used similar expressions, such as \textit{Pluralitas non est ponenda sine necessitate}.”

\textsuperscript{4}Levinas (2001; p. 44) commenting on Merleau-Ponty and Sartre in \textit{Is it righteous to be?} (Stanford University Press).
1 The Problematic: An Introduction

In a 1999 retrospective reflection on his contributions to “equilibrium, trade and capital accumulation”, Lionel McKenzie presents his 1957 paper on “demand theory without a utility index” as a digression, a three-paragraph intrusion into his main narrative.

I may be forgiven for intruding here a contribution to demand theory which was not itself a major advance but which led to some developments in the hands of Leo Hurwicz and his collaborators. Hurwicz was in the audience when my paper was presented to the Econometric Society. Later Hurwicz (1971) and Hurwicz and Uzawa (1971) used this approach to demand theory to do some definitive work on the old problem of integrability, the problem that asks when a field of indifference directions can be integrated into a set of indifference curves.

In prose that is reminiscent of Ramanujan’s letter to Hardy, McKenzie describes his then forty-year old paper not as a major advance in itself but simply as an alternative approach; another proof of well-known results; a method to be used by others to do definitive work on a longstanding problem; a contribution to be singled out perhaps more for its unintended consequences rather than for what it itself set out to achieve. The irony that an approach designed to show the dispensability of a utility index helped resurrect it only underscores these observations, and seemingly offers yet another instance of what Samuelson (1974, p. 1266) calls the “serendipity of irrelevant issues”.

As so often happens in the history of scientific thought, one gains attention and appreciation for one’s innovations for reasons that prove to have been irrelevancies.

However, the matter does not end here. In a profession not overly-inclined to burden its subject by semantic issues and terminological disputes, McKenzie’s emphasis on a proper naming is of interest.

Today the minimum income function is often referred to as the minimum cost function or the minimum expenditure function, sometimes with the qualifier “minimum” omitted. I do not regard this change of terminology as an improvement.

It is of course the minimum income function that is introduced in the very first section of the first chapter on the theory of demand in McKenzie’s 2002 text on Classical General Equilibrium Theory. And not unlike Hicks’ treatment in his 1939 work, the mathematical properties of this construct are relegated to a seven-part Appendix. The first substantive paragraph of the book sights Walras’ 1874-77 formulation as furnishing the “classical paradigm”, and the 1957 paper as introducing “a method” by which the results of Hicks (1939) and Samuelson (1947) could be derived.

We call this the direct method and this is the approach that we will treat as primary.

We shall have more to say of McKenzie’s own fast-paced, 2002 survey of 20th-century demand theory in the sequel; here we content ourselves simply by listing the trajectory

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5 This ironic thrust, and Samuelson’s phrase and the quotation below, can be seen as the driving engines of this essay, reaching their culmination in our engagement with Stigler’s methodological imperatives in Section 4 below. In particular, we refer the reader to Sections 3.1-3.3, and to the Appendix. They also feeds into our sighting of a classical text by it saying more than it says; see Footnotes 14, 90 and 95.

6 But an Appendix that follows the Chapter rather than the substance of the entire book.
of names and dates that McKenzie uses to outline it. The classification of the subject matter is now clear: the direct method; demand theory without transitivity; the classical method involving Roy’s identity, along with the recovery of a smooth utility function from preferences; the weak axiom of revealed preference along with a recovery of the preference relation from demand; and market demand functions. Yet, even with these placements, by the author and within two years of each other, perhaps despite them, the question remains: what did a paper sighted several times as pioneering precisely pioneer? How did the method, and the construct that underlies it, reorient the standard line of inquiry? which lines did it close, and which new ones did it open? In so far as a method can be distinguished from a construct, are there constructs that, once introduced, rupture the very rubric that they were initially designed to investigate? More generally, what does novelty mean in this context? and how did the contribution of 1957 evolve from the first 1956 attempt, and then lead to the novel view of all that was within its purview? and in so doing, opened other views?

These questions, and the problematic they delineate, structure this essay into what we see to be two complementary parts. For one, we learn for ourselves, and recall for the reader, how the minimum income function feeds into the high points of 20th-century demand theory: in particular, into discussion of consumer’s surplus, compensation criteria, index numbers, complementarity, duality, and general equilibrium theory. By necessity, this part of the essay is of a summarizing and classificatory nature, a quick laying down of trajectories: evidentiary material in open view, to see if it conforms to the patterns that are conventionally imposed on it. This is, at one level, the substantive bulk of the essay; but at another possibly more philosophical level, this material is simply input for a motivation that at first glance seems to be an epistemological and

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7In order of appearance, Walras, Hicks and Samuelson are followed by Sonnenschein (1971), Debreu (1954, 1972), Mitushin-Polterovich in Mas-Colell (1991), Roy (1947), Wald (1934), Gale (1960), Hurwicz-Uzawa (1971), Diewert (1977), Hildenbrand (1983), McKenzie (1954) and Debreu (1970). In the Appendices, Diewert and Browning-Chiappori (1998) follow Uzawa (1971). It should also be noted that other than appeal to the mathematical results of a 1911 calculus text of Wilson’s, Berge (1963), Alexandroff (1939), Rockafellar (1960) and Fenchel (1953) are the sole mathematical references for these Appendices. Since they are McKenzie’s references, we do not cite them in our bibliography.

8Since this essay has to draw inevitably on some subtle points of demand theory, we make a special effort to send the non-expert reader to Kreps (2013, Chapter 10), and to the earlier treatments of Mas-Colell-Whinston-Green (1995, Chapter 3), Deaton (1986), Takayama (1985), Katzner (1970) and Morishima et al. (1973). The fact that the Kreps, Deaton, Katzner and Morishima texts do not reference McKenzie (1957) may be an advantage in this regard.

9Most recently by the Spanish scholars Martinez-Legas-Santos (1996, p. 159) who write “We now show some simple generalizations of certain classical results, concluding with a derivation of Slutsky’s equation as in the pioneering work of McKenzie (1957).” Presenting his own results as a “completely novel approach to the comparative statics of demand,” Willig (1976b) emphasizes that it is “based entirely on the income compensation function, pioneered by McKenzie [5] and Hurwicz and Uzawa [3].” Willig’s reference to McKenzie is subsequently dropped, and, in Willig (1978), Hurwicz-Uzawa (1976b) is the sole reference. Also see Willig (1979), taking care to distinguish G. McKenzie from L. W. McKenzie.

10At the very outset, we flag for the reader the importance of this word for this essay. We borrow it from the history of art to designate not a single, specific and clearly-specified, problem within a field, but to get a larger sense of the problem the field itself addresses from a substantive, conceptual and procedural points of view. Thus, rather than an understanding of a single painting, it is the placing of the painting in the problematic of “absorption and theatricality” in the 18th-century European painting, as in the work of Michael Fried, or the problematic of “authenticity” in Western art tradition as in Pippin (2005), “Authenticity in painting: remarks of Michael Fried’s art history,” Critical Inquiry 31:575-598. We shall be returning to this footnote as we elucidate and elaborate the problematic of 20th-century demand theory in the light of McKenzie’s 1956-57 paper.

11We are indebted to an anonymous referee for his suggestion that this material “aid the reader to brush upon their demand theory knowledge (a recollection from Ph. D. courses for many),” and that therefore “it would be useful for many of the readers to go through the algebra of the main results in more details.”
historiographic one: to use McKenzie’s 1957 contribution as an instance for reflection on how what is 20\textsuperscript{th}-century demand theory, and how is one to write its history? But a less facile, and a more careful reading, will reveal that this is an investigation on how to model what it is to model, and therefore by theorizing about the doing of theory, it is squarely within the domain of theory. It is thus not an essay\textsuperscript{12} on the history of economic thought: chapters and texts still being written on consumer theory and authors still trying to digest, with or without acknowledgement, what McKenzie accomplished in 1957, the salient features on 20\textsuperscript{th}-century demand theory. It is not only how this theory ought to be exposed and presented, a question of pedagogy, but also how the consumer is to be understood and theorized.

As such, this is a plea – a programmatic plea at least to the microeconomics part of the profession, the macroeconomics part has already accepted this pleading after their most recent humbling in the 2007 financial crisis – that a model, and its theorems, has a history, a problematic to which it is a response, and that it cannot be understood, much less extended and built upon, without keeping that problematic firmly in view.\textsuperscript{13} To be sure, time-series analysts have always intimately understood that a neglect of past observations – or the past more generally – is a neglect for which one pays, but what we attempt to bring out in this essay goes beyond this. It is not the efficiency cost, a trade-off at the margin that can be asymptotically remedied with more observations or a longer time-horizon or with more and better data – but rather a cost of misdirection, of losing one’s way, of going down blind alleys, a chasing of the will-o’-wisps, of engaging in theorizing that is blind, one that leads to a blinding of a community, rather than giving it sight and insight. Thus, we use McKenzie’s 1957, intrusion or breakthrough (it does not quite matter in this context), to reflect on current modeling practices: into how the “economics’ community” sights its classical texts and uses them to constitute its canon; how it gives meaning to the terms “classical, definitive, seminal, landmark, breakthrough, useful” and differentiates them from the “derivative, and the mere extensions, technical or otherwise”; the role that it gives to citations.

Without understanding a breakthrough, one can hardly make a breakthrough; without understanding what is original, how can one evaluate originality, leaving aside the obvious question of being oneself original? But this striving for a deeper understanding is hardly easy, and our conclusion, based on reading McKenzie (1956-57), shall be that classic always says more than it means to say, that it is always under-determined and therefore it is essential to cultivate good habits of reading.\textsuperscript{14} This perforce takes us to Stigler’s (1965) distinction between “textual and scientific exegesis”.\textsuperscript{15} Without

\textsuperscript{12}The fact that it is not scarcely needs a mention, much less an emphatic one. An essay on the history of economic thought, even limited to 20\textsuperscript{th}-century demand theory would need, at the very least, to consider the papers of Samuelson, Georgescu-Roegen, Roy Allen, Hicks and undoubtedly many others. As regards texts still being written, see the relevant chapters in, for example, Kreps (2013).

\textsuperscript{13}To underscore Footnote 10 on the use of the word “problematic” in this essay, it is precisely the framing of a theorem in a larger context, rather than its explication, that we have in mind. It is what elsewhere we call second-order as opposed to first-order theory.

\textsuperscript{14}See Footnote 95 below and the text it footnotes. Also note in this we depart from Quentin Skinner and his emphasis on intentions; see Tully-Skinner (1989).

\textsuperscript{15}We remind the reader that in Stigler (1949), the fourth lecture is titled \textit{The mathematical method in economics}. As we shall have occasion to emphasize below, Stigler was a skilled practitioner of second-order theory among distinguished
an intimate understanding of what we are referring here to the “the problematic”, one can hardly determine what to include and what to exclude? what to attempt and what to avoid? what to theorize for and what to theorize against? and most starkly put, how to do theory?¹⁶ As such, this meta-theoretical investigation relating to theorizing, and to historicizing, currently considered in some quarters as a softer, even feminine, mode of expression, constitutes our principal motivation. We move away from Stigler when we make the argument that McKenzie’s intrusion, properly understood, changed the very nature of what it was intruding into.¹⁷ Thus, in addition to being of interest in themselves, the intermediate sections can also be seen as footholds, though not mere grist for the methodological mill, for the second level of this investigation taken up in the last section of this essay, and which are intended to give the argument its completion.

2 A Textual Exegesis

In 1999, McKenzie summarizes his 1957 contribution, evaluates further what he has already seen then as not being an advance in itself.

I showed how the Slutsky equation for the separation of the effect of price change on demand between a pure substitution effect and an income effect could be derived by the use of the properties of the minimum income function. The minimum income is the smallest income, given prices, that allows a consumer to reach a given level of preference, determined by a reference consumption bundle.

The question McKenzie answers goes some way in the history of the subject. Samuelson-Swamy (1974, Footnote 3) furnish the following quote from a 1949 monograph on index numbers.¹⁸

Writing in 1707, William Fleetwood, Bishop of Ely, set himself the task of determining the relative difference in money income which would provide for the student at the university of Oxford ‘the same Ease and Favour’ in his day and 260 years before. The fifteenth century B.C. Indian treatise Arthastra shows similar concerns.

In his evaluation, McKenzie focuses on his observation that the minimum income function is concave in prices, and that its first derivative, when it exists, yields the compensated demand function. Everything else follows from what was even then standard theory.¹⁹ We reproduce the entire second of the three paragraphs.

I wrote this function as \( M_x(p) \). It is illustrated in Figure 3.²⁰ The matrix of second partial derivatives of \( M_x(p) \) (the Hessian matrix) is equal to the substitution matrix (or to the matrix of first derivatives of the compensated demand functions (the Jacobian of those functions)). Since \( M_x(p) \) is a concave

(footnotes)

¹⁶This is a long-standing research program of one of the authors, and this essay can be seen as its ongoing concern with second-order theory; see the conclusion of Khan (1993a) and also Khan (1993b, 2003, 2014) for related concerns. In this connection, also see Footnotes 10 and 13

¹⁷As Jacobus (2012; p. 66) writes in Romantic Things: a Tree, a Rock, a Cloud (Chicago: Chicago University Press), the “intrusive, the intruder, or intrusion marks the threshold of what is living as well as [its] ... otherness.” It was 20th-century demand theory that McKenzie was intruding upon.

¹⁸See M. J. Ulmer The Economic Theory of Cost of Living index Numbers, p. 28.

¹⁹For the current state of the art, we send the reader and/or a budding expert of 20th-century demand theory to the references in Footnote 8.

²⁰Rather than the reproduction of Figure 3 in McKenzie (1999), we choose to reproduce as Figure 2 of this essay, a diagram in McKenzie (2002). In any case, Figure 3 of this essay illustrates the function \( M_x(p) \).
function, this matrix is negative semi-definite, from which the results of pure-demand theory follow. Unlike the classical approach, this derivation of demand theory avoids the use of mathematics of determinants and Jacobi’s theorem. Also, the results do not depend on goods being divisible and or utility being a quasi-concave function. Since \( M_x(p) \) is concave, it has a well-defined Hessian at almost every point in the price space.

At one level, this is all there is to it. But given the theoretical and epistemological thrust of this essay, this paragraph could usefully be supplemented with an exegesis of the original texts.

2.1 Reading Lionel McKenzie

It is best to begin with the eleven paragraphs of McKenzie (1956) titled *A Direct Approach to the Slutsky Relation*. It begins as follows:

> My purpose is to give a discussion of the Slutsky relation of demand theory without defining a utility function. I shall depend on the properties of convex sets. The foundations for a treatment of demand theory by convex set methods have been laid by Arrow and Debreu (1954). This approach is related to the revealed preference analysis of Samuelson (1947) and to the analysis which depends on the indirect utility function. The latter has been used recently by Houthakker (1952) to give a proof of the Slutsky relation akin to ours. However, I shall offer a straightforward discussion that eschews the utility function.

The introductory paragraph begins and ends by underscoring the motivation behind the exercise: to rid the Slutsky decomposition, and thereby 20th-century demand theory, of the utility function.

We shall not reproduce, but merely refer the reader to, Samuelson’s masterful 1974 evaluation of Slutsky’s contribution to 20th-century demand theory; his Footnote 4, however, is directly relevant to the exegesis that we are in the process of working through.

Of course, what you find in Slutsky (1915) and Hicks-Allen (1934) is something much more important than palaver about complementarity – namely recognition of the canonical simplicity demand analysis assumes when *ceteris paribus* terms \( \partial x_i / \partial p_i, \partial x_i / \partial p_j, \partial \text{income} \) are used to form “compensated” compounds, \( \partial x_i / \partial p_j + x_j(\partial x_i / \partial \text{income}) \).

After the introductory paragraph spelling out his purpose, and quoted above, McKenzie (1956) turns, following Samuelson (1947) and Hicks (1939), to a definition of the Slutsky

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21It is worth pointing out in this connection that Afriat (1972) refers to the 1956 CFDU (Cowles Foundation Discussion Paper) rather than the 1957 published version. Given his (and also McKenzie’s 2002) emphasis on McKenzie’s method, the title of Afriat (1980, Chapter 2), this is not altogether surprising.

22A footnote also cites Georgescu-Roegen (1954) for his “discussion of preference orderings.” In the sequel, we shall use the convention to reference a text in terms of the dates provided in the bibliography of this essay regardless of the way it is cited in a quote from another text.

23McKenzie defines the indirect utility function in a footnote. Note also that our convention, as specified in Footnote 22 above, leads us now to change McKenzie’s 1951 dating of Houthakker’s paper to Houthakker (1952).

24See the two-paragraphed seventh subsection in the historical review in Samuelson (1974, pp. 1282-1282). Also see Chipman-Lenfant (2002). It is worth pointing, and warning the reader of this (our) bias: the first version of this paper was rejected by Professors Mirowski and Hands for inclusion in the HOPE (History of Political Economy) volume of the Conference partly on the grounds of this bias towards, or at least emphasis of, the work of Samuelson’s.

25It may be worth noting that McKenzie (1956, 1957) does not allow himself to be diverted by the substitute/complementary distinction in his exposition; the words do not appear in the texts.
relation and to the laying out of the notation for these “compensated” compounds.\textsuperscript{26}

\[ s_{ij} \equiv \frac{\delta x_i}{\delta p_j} \equiv \frac{\partial x_i}{\partial p_j} + k_i \frac{\partial x_i}{\partial m} = \frac{\partial x_j}{\partial p_i} + k_i \frac{\partial x_i}{\partial m} = \frac{\delta x_i}{\delta p_j} \equiv s_{ji}. \] \hspace{1cm} (1)

Note that in his usage of \( k_i, k_j \) rather than \( x_i, x_j \), McKenzie thus initially leaves open the precise compensation that is required; it is in the last numbered equation\textsuperscript{27} of the paper that this equality is established with the \( x_i, x_j \) terms. Indeed, in the third paragraph, the notion of a support function to a convex set is defined and the claim laid out.

I shall show that \( \frac{\delta x_i}{\delta p_j} \) and \( \frac{\delta x_j}{\delta p_i} \) are second order partial derivatives of the support function of a certain convex set, and therefore equal whenever they exist.

McKenzie observes that under the assumption of closedness of \( \succeq \), “the set \( C_{\bar{x}} \) of combinations\textsuperscript{28} of goods indifferent to \( \bar{x} \) will be closed.”\textsuperscript{29} Under a local non-satiation assumption on preferences, it is shown that for a preference maximizing bundle \( \bar{x} \), whose existence at prices \( p \) and income \( M \) is presumed,

\[ p\bar{x} = M_\bar{x}(p), \text{ where } M_\bar{x}(p) \equiv ph(p, \bar{x}) \leq px \text{ for all } x \in C_{\bar{x}} = \{ x \in S : x \succeq \bar{x} \}. \] \hspace{1cm} (2)

The function \( M_\bar{x}(p) \) is the minimum income function. It differs from the conventional textbook expenditure function by substituting a reference bundle \( \bar{x} \) for a reference utility level \( \bar{u} \), the very thing to be dispensed with.\textsuperscript{30} The last four paragraphs then establish with some dispatch what needs to be established. In this, several points deserve emphasis:

(i) Fenchel’s 1953 notes are invoked, with three page numbers, for the assertion that first and second derivatives of a convex function exist “almost everywhere,”

(ii) the notion of “almost everywhere” in Euclidean \( n \)-space is formally defined in a footnote,

(iii) the importance of the “minimum cost combination [being] unique in the neighborhood of \( p \)” is noted and McKenzie (1955) referred to for a sufficient condition,

(iv) the fact that \( p(\delta x/\delta p_i) \) is zero is proved in the ninth paragraph of the paper.

\subsection*{2.2 A Modern Restatement}

We digress to illustrate how (i)-(iv) can be used to derive what McKenzie calls the Slutsky relation. For the convenience of the reader, we bring McKenzie’s derivation forward to the present. The function \( M_\bar{x}(p) \) in (2) is the minimum income function, just described. To preface the argument, completeness, transitivity, and continuity of preferences assure existence of solutions to the preference maximization and income minimization problems.

\textsuperscript{26}The following supplements McKenzie’s first numbered equation by the less cumbersome \( s_{ij} \) notation of current usage by Afriat (1972, Equation 3.1; 1993) and others. In line with Footnotes 8 and 19, the non-expert reader is again referred to the relevant chapters in Kreps (2013).

\textsuperscript{27}There is a typographical error in McKenzie (1956) in that this is the fifth rather than the fourth numbered equation in the paper.

\textsuperscript{28}It is of literary interest that the word “batch” or “combination” is used instead of the later terminology of a “bundle; also see Newman-Reed (1958) for reliance on “batch”, and anticipating somewhat, on “strips.”

\textsuperscript{29}As will be indicated, this sentence is of consequence to discussion subsequent to the paper.

\textsuperscript{30}We are indebted to an anonymous referee for his suggested emphasis of this point.
When added to this ensemble of assumptions, local nonsatiation and the cheaper-point assumption—formally defined in the Appendix—ensure equality between the preference maximizing and income minimizing choices:

\[ f_i(p, M_x(p)) = h_i(p, x). \]  

for every commodity \( i \). Point (i) above already ensures that \( M_x(p) \) is twice differentiable at ‘almost every’ price. Let \( p \) be one such price. By the envelope theorem\(^{31}\) and (3)

\[ \frac{\partial M_x(p)}{\partial p_i} = h_i(p, x) = f_i(p, M_x(p)), \]  

so both the preference maximizing and income minimizing choices are unique. Since \( M_x(\cdot) \) is twice differentiable at \( p \), the cross partial derivatives of \( M \) are symmetric:

\[ \frac{\partial^2 M_x(p)}{\partial p_i \partial p_j} = \frac{\partial^2 M_x(p)}{\partial p_j \partial p_i}. \]  

Differentiate (4) with respect to \( p_j \), insert into (5) to get (1) with \( k_i = x_i \). And that is all there is to it. Contrast the directness of this argument with the then-standard derivation found in, for example, Hicks’s *Value and Capital* (1946, Appendix to Chapters II and III), a derivation presented as standard as late as Katzner (1970), Section 3.4. The older derivation starts from the first order conditions from the utility maximization and income minimization problems, invokes the implicit function theorem to grind out price and income derivatives of the demands, and then rearranges the result to arrive at (1).\(^{32}\)

It is also worth highlighting that McKenzie (1956) calls equation (1) the Slutsky relation; it asserts the symmetry of the Slutsky substitution terms. But by common usage then and now the Slutsky equation is the decomposition of the effect of price changes on the ordinary demands into substitution and income effects:

\[ \frac{\partial f_i(p, m)}{\partial p_j} = \frac{\partial h_i(p, x)}{\partial p_j} + f_j \frac{\partial f_i(p, m)}{\partial m}, \]  

where \( m \) is evaluated at \( M_x(p) \). It turns out that on his way to (1), McKenzie passes through (6), as will any reader who carries out the derivation of (1) just outlined.\(^{33}\)

Finally, we point out that the importance of item (iv) in Section 2.1 is that McKenzie uses it to establish what we now refer to as the envelope theorem for the income minimization problem, a result he proves without fanfare, as a matter of course. This fact supports Takayama’s (1985, p. 140) usage in referring to the first equality in (4) as the McKenzie-Shephard lemma, rather than just Shephard’s lemma.

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\(^{31}\)By “the” envelope theorem we mean not of course the classic version which assumes smoothness of an objective function and of solutions, but, for example, Theorem 1 in Milgrom and Segal (2002).

\(^{32}\)Varian (1978) marks the transition in graduate texts from the older methods to one close to McKenzie’s (though using a reference utility rather than a reference bundle to define the expenditure function).

\(^{33}\)McKenzie (2002, p. 12) is conscious of his practice, and citing Samuelson’s *Foundations*, writes “The decomposition is called the Slutsky relation for its discoverer.”
A fundamental question, for both analytical and historical reasons, is how the seventeen paragraphs of McKenzie (1957) go beyond the eleven of McKenzie (1956). Ignoring typographical errors, there are the inevitable questions of references: their number remains unchanged but Georgescu-Roegen (1954) is substituted by Hicks (1956) and the paper’s independence of it noted in the acknowledgement. In addition to epistemological criteria—mathematical form and brevity—this acknowledgement also notes “The argument was also partly anticipated by Allen (1950).” In terms of substantial modifications of the 1956 paper, however, there are at least four.

(i) Both the new title and the first paragraph signal a new broader ambition: no longer “An Approach to the Slutsky Relation” the title is now “Demand Theory without a Utility Index.” The 1957 opens thus.

The modern revolution in the theory of demand has been to replace utility as a measurable quantity with a utility index which is arbitrary up to a strictly monotonic transformation. It is my purpose here to describe an approach to the theory of demand which dispenses with the utility index entirely. This use of Occam’s razor does not, however, complicate the derivation of the major propositions of demand theory, but rather, in at least the case of Slutsky’s equation, leads to an important simplification.

(ii) The conclusions themselves are more ambitious: he derives not just the Slutsky relation, but the testable implications of demand theory itself. In the four concluding paragraphs—three of them new in the 1957 revision—McKenzie deduces all but one of these: he proves that the Slutsky matrix, composed of the $s_{ij}$ terms defined above in equation (1), is symmetric and negative semi-definite, the last property sometimes called the compensated law of demand.

(iii) The argument is logically tighter: in 1956, he omits the assumption that the consumption set is closed, and mistakenly asserts that the existence of cross partial derivatives implies their equality. The notation $M_x(p)$ and the terminology of a “minimum income function” is introduced in the fifth paragraph; in 1956 this function is merely referred to as a support function. In the seventh to tenth paragraphs, McKenzie explicitly provides for two things that enable his demonstration: local non-satiation of preferences so that the preference maximizing combination is on the boundary of the budget set, $pf(p,m) = m$, and the existence of a “locally cheaper point” in the budget set that guarantees that there is indeed a combination at which the consumer is indifferent to the original combination. McKenzie notes:

[Neither] of these requirements represents a limitation on the analysis, for they are needed to make the Slutsky equation meaningful in any case.

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34 The acknowledgement footnote reads: “This paper was first written independently, but it has since benefitted from the very similar approach in Professor Hicks’ new book (1956). The chief virtues of the presentation here are perhaps the mathematical form and brevity.” It is of interest, however, that there is no reference to this book in McKenzie (2002).

35 It is interesting that this article disappears from the literature: it does not find a place in the references of, for example, Fernandez-Grela (2005), Lenfant (2006), Samuelson (1974) or Samuelson-Swamy (1974). The first papers were presented at the Duke Conference, and this is perhaps the place to acknowledge our indebtedness to them.

36 This is not the complete first paragraph; it is then completed along the lines of the first paragraph of 1956.

37 The other implication, which he merely alludes to in the last sentence of the paper, is homogeneity of degree 0 in prices and income; see Mas-Colell, Whinston, and Green (1995, p. 30-34) and Kreps (2013, p. 281).
In other words, if either of these conditions are not fulfilled, there is nothing to prove and we may as well stay at home!

(iv) A new diagram, reproduced as Figure 1 in this essay, is introduced: it substantiates the three points mentioned above. More to the point, it makes clear that McKenzie (1957) dispenses with the 1956 assumption of a convex consumption set, and therefore of convex preferences as well. This is of obvious importance to subsequent work.

In a subsequent comment to the demurral of Newman-Reed (1956) that McKenzie has dispensed with the utility index entirely, McKenzie (1958) makes the further point that:

The assumption of closure is inessential to my argument and to demand theory, one merely redefines the notion of compensation, so that the consumer is compensated if he receives the greatest lower bound of incomes which permit him to reach a position at least as good as he could achieve under the initial conditions.

This is to say that the minimum operator in the definition of the minimum income function can be replaced by the infimum operator as in Equations (2.4) below. Furthermore, McKenzie notes that since the “second differential exists for almost every \( p \), a unique best combination of goods exists for almost every \( p \) and income \( M_x(p) \). What McKenzie passes over in silence is that, for a fixed \( p \), the minimum income function is itself a utility index. McKenzie (1957, p. 188) already pointed to this possibility with the observation that “\( M_x(p) \) [is constant] for movements along the indifference locus of \( x \).” We confirm in an Appendix that a proof requires very little beyond what is in already in the 1957 paper.

In conclusion to this section, we turn to refer to some selected recent literature. First Karlin’s 1959 judgement:

The modern theory of consumption originated with Walras and Jevons. Slutsky and later Hicks greatly elaborated this theory. Slutsky’s analysis relies on the existence of a utility indicator function, which is used in deducing the the Slutsky relations satisfied by the demand functions. McKenzie derived the same conclusions without the use of a utility indicator function. Our proof is a modification of McKenzie’s.\(^{38}\)

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\(^{38}\)The third footnote explicitly states “The basis for the proof is illustrated in the figure.” This is perhaps also the place to mention that the eight footnotes of 1956 are replaced by three, the second invoking Young’s theorem and referencing La Vallée Poussin’s book on infinitesimal analysis, and the second referring to indivisible commodities, about which a little more below.

\(^{39}\)In this sense, this has been a journey somewhat similar to that of Khan (1993, p. 5), “I refer you to Figure 1 which is reproduced from McKenzie and gives a transparent “bird’s eye” view of the basic ideas of the proof.” It also allows an easy comparison with subsequent work, as the diagram here does as well; see Section 3.10 below.

\(^{40}\)We shall see the full importance of these considerations emerging in the work on compensated equilibria in general equilibrium to be discussed below in Section 4.8.

\(^{41}\)McKenzie writes, “My discussant, Frank Hahn, and I independently noted this point when my paper was first presented to the Econometric Society.”

\(^{42}\)A limitation of the theorem – as a piece of economic theory, rather than as an observation pertaining to its history – is that the domain of the minimum income function is restricted to the set of consumption points that satisfy a cheaper-point assumption.

\(^{43}\)See Karlin (1959, p. 290). I do not reproduce Karlin’s text exactly. He gives references to the (classic!) paper of Slutsky and to Hicks (1939) but not to Walras and Jevons. In particular, he identifies his own theorem as Slutsky’s.
For a reader interested in seeing how far the subject has progressed beyond Karlin (1959), she may look at Böhm-Haller (1987), Hatta (1987), and in particular Martínez-Legaz-Santos (1996, Section 4.3). We move on – beyond the exegesis of these eleven (subsequently expanded to seventeen) paragraphs.

2.4 Two Precursors

1953 was an important year for the subject matter of this essay: it saw the publication of Ronald Shephard’s monograph on Cost and Production Functions as well as Yokoyama’s paper on a Logical Foundation of the Theory of Consumer’s Demand.

In his 1981 Forward, Jorgenson ranks Shephard’s volume as “one of the most original contributions to economic theory of all time” and sees its contribution under three separate headings:

(i) the duality between cost and production “based on convex analysis and completely modern in its orientation,”

(ii) Shephard’s lemma which has had such “far-reaching influence on applied econometrics over the past decade,”

(iii) homotheticity and homothetic separability, of “critical importance ... in the theory of aggregation and index numbers.”

In the 1953 volume itself, Shephard is silent about the application of his work to the theory of the consumer, but in the fuller 1970 version, Shephard (1970), he devotes an Appendix to Utility Functions and begins it with the statement:

The theory developed in the chapters for the production function is readily adapted to utility functions.

In less than four and half pages, he translates his postulates on the set of production plans to that on a set of consumption plans, derives the indirect utility function and deduces what is now called Roy’s identity. He shows how the “direct and indirect utility functions are dually determined from each other” if the graph of the utility function is convex. There is however, no discussion of the minimum expenditure function, and neither Roy (1931, 1942) nor McKenzie (1957) make it to Shephard’s thirty-one references.

The second precursor we identify, and seek to canonize, is explicitly introduced in Sakai (1977), a paper based on his Rochester Ph. D. dissertation. In his Footnote 6, Sakai observes the following.

Income compensation functions were first introduced by McKenzie (1957) and Yokoyama (1953), independently, in terms of preference orderings on the commodity space.

This statement is indeed true; the only question is to delineate more fully the sense in which it is true. Yokoyama begins his remarkable paper of seven sections, with a complete, transitive, monotonic relation $\succeq$ on a set $\mathcal{M}$, and then derives the two relations, “indifferent to” $\sim$, and “preferable to” $\succ$. This allows him to define $\min_{y \in I(x)} p y =$

\footnote{We refer the reader via these papers to the relevant chapters referenced in Footnote 8 on standard microeconomic theory of the consumer.}
\[ \arg \min_{y \in I(x)} py = px^* = M^* \] where \( I(x) = \{ y \in M : y \sim x \} \). Under the assumption that “the consumer’s demand is a one-valued function” of prices and income, Yokoyama observes that the uncompensated demand at prices \( p \) and income \( M^* \) is precisely \( x^* \). However, what Yokoyama’s very notation makes explicit, his concern is not with exploiting the dependence of \( M^* \) on \( p \) or, as we shall see in the sequel, on \( x \). As he writes in the context of his postulates on the preference relation in the first section of his paper (entitled The Choice):

When we want to take the interpretation of the substitution and income effects given by Prof. Slutsky, the above conditions are quite sufficient to develop the theory of consumer’s demand. However, when we want to take the interpretation of the substitution and income effects given by Prof. Hicks, we must add one more condition.

It is precisely the distinction between these two different, both fruitful, ways of decomposing uncompensated demands that is Yokoyama’s primary concern. He derives these two types of decompositions for finite differences of prices, and examines the limiting situation using the language of differential geometry. Rather than projecting it onto McKenzie’s work, we would suggest that the appropriate context for a fuller appreciation of the paper is Mosak (1942), with its famous footnote by Abraham Wald, and Afriat (1980; Chapter 1). Vartia (1983, pp. 82-83) writes:

As demonstrated first by Hicks and Samuelson and shown later by Shephard, L. McKenzie, Diewert, Afriat and others, it is possible to define the minimum expenditure (or cost) function ... under fairly general conditions on \( u(x) \). \( M_x(p) \) is the minimum expenditure needed to buy the well-being determined by \( x \) (i.e. some \( \tilde{x} \) indifferent to \( x \)) when prices are \( p \). For any given \( p \) the function \( M_x(p) \) is a utility function, in particular \( x \sim \tilde{x} \iff M_x(p) = M_{\tilde{x}}(p) \) (see e.g., Afriat). If \( x = f(p, M) \) or \( (p, x) \) is an equilibrium pair, then \( M = px = M_x(p) \). We will use \( M_x(p) \) freely in our later operations.\(^{45}\)

The point of course is that \( M_x(p) \) is independent of a utility function, and for given prices \( p \), is itself a utility function.

### 2.5 An Evaluative Summary

We conclude this long section with our view of McKenzie’s contribution to mid-century demand theory. As pointed out in Section 2.3(i), McKenzie saw his 1957 work as continuing the “revolution” in demand theory that replaced cardinal with ordinal utility by replacing ordinal utility with a preference relation. What his brief narrative leaves out is that the non-integrability cases of Allen-Hicks (1934) and Georgescu-Roegen (1936) are already incompatible with ordinal utility; that the aim of Samuelson’s (1938) revealed preference approach had been precisely to dispose not just of utility, but even of preferences; that the combined forces of Samuelson (1938) and Houthakker (1950) had already produced a complete theory free of both utility and preferences, and one logically equivalent to one based on cardinal or ordinal utility.

If McKenzie’s novelty is not simply Demand Theory without a Utility Index, then what is it? The expenditure or cost function had already been taken advantage of by

\(^{45}\)We take several kinds of liberties with the notation of this text.
Shephard (1953) and Hicks (1956), with commodity demands derived simply by differentiating this function (as McKenzie (1956) already confirmed). This of course contrasts with the classical approach, which derives demand from utility (or output in the case of production). The novelty of the minimum income function is the use of a reference bundle instead of a reference utility level as the second argument of the function.\textsuperscript{46} That the demands are the derivative of the minimum income function allowed for an almost effortless derivation of them from preferences rather than utility. In addition, what McKenzie realized between the 1956 and 1957 versions of the paper is he could drop the assumption that preferences are convex, since the concavity of the minimal income function in prices follows for an arbitrary consumption set and preference relation on it.\textsuperscript{47}

This much his approach, but what of his results? In the 1956 draft, he had already derived the Slutsky symmetry property that had eluded Samuelson (1938) who had imposed only budget balance and the weak axiom of revealed preference. Houthakker’s (1950) strong axiom finally allowed a deduction of the symmetry property, but at the cost of a long argument. What McKenzie’s (1957) method did was to allowed was a dramatically streamlined derivation of all the testable implications of demand theory. Furthermore, as we have noted, McKenize 1956-7 emphasized the price argument in the minimum income function, as his “$M_x(p)$” notation makes clear. What remained unexploited in 1957 were the implications of this function when viewed as a function of the bundle $x$. As we will emphasize in the next Section, this aspect of the minimum income function would lay the foundations for both consumer theory and general equilibrium theory with nonconvex, nontransitive preferences, for welfare economics in the form of money-metric utility, and somewhat ironically, for the integrability problem of deriving utility from demand.\textsuperscript{48}

### 3 20\textsuperscript{th}-Century Demand Theory: A Review

We maintain McKenzie’s 1999 stance to demand theory, and undertake the review of the subject as outsiders to the subject, as modelers going beyond their model, as experts outside the spatio-temporal range of their expertise, so to speak. We are interested primarily in the impact of the minimum income function. This interest often forces us to close a topic and move to another as soon as it gets interesting. If we make connections that are of potential interest, or indicate lacunae, all the better, but this is not our primary motivation.

\textsuperscript{46}Shepard’s (1953) cost function is mathematically equivalent to what we know call an expenditure function, with his reference output level playing the role of a reference utility. In his 1970 revision, Shepard replaces a reference output with a reference input requirement set, formally equivalent to the minimum income function applied to production.

\textsuperscript{47}The concavity of the minimal income function in prices is obtained “for free”. In particular and somewhat ironically, as we shall emphasize later in Section 3.3, it enables a derivation of a utility function itself. Also, see Footnote 96.

\textsuperscript{48}Footnote 34 already alludes to the elimination of Hicks (1956) from his 2002 treatment of consumer theory: his eleven references to Hicks in his index all pertain to Value and Capital. This is to us entirely justified, but see Footnote 79 below. We are grateful to an anonymous referee for suggesting this subsection.
3.1 Additional Results in Demand Theory

After Hicks’ Revision of 1956, and the Samuelson-Gorman 1968-69 investigations of additive separability, 49 Robert Willig’s (1976) five results on the elasticities of demand of an individual consumer surely deserve prominence. After a summary presentation, he writes with some surety:

These theorems seem to have escaped the notice of fifty years of demand theory (391).

He notes that his results all involve “second derivatives of neoclassical demand functions,” and conjectures that the reason they have been missed may lie possibly in the fact that “our standard comparative statics methodology has only been well developed to analyze first derivatives,” and an available alternative methodology had not been used.

In this paper, a completely novel approach to the comparative statics of demand is taken. It is based entirely on the income compensation function, pioneered by McKenzie (1957) and Hurwicz and Uzawa (1971). 50

So what is this “completely novel approach” that is “based entirely” on $M_x(p)$?

In a section titled Basic Tools, Willig begins with the standard theory of the consumer: a utility index which, given strictly positive prices $p$ and a positive income level $m$, furnishes him with the following two standard objects, the Marshallian (uncompensated) demand function and the indirect utility function:

$$f(p, m) = \arg \max_{px \leq m} U(x) \quad \text{and} \quad V(p, m) = U(f(p, m)).$$

Willig can now give an implicit definition of the minimum income function as the “income necessary with prices $q$ to enable the optimizing consumer to reach the same level of utility as that obtained with prices $p$ and income $m$”; see Figure 3 for the equality

$$V(q, M_f(p, m)(q)) = V(p, m) \text{ for all } m > 0 \text{ and } q >> 0.$$

Since the indirect utility function is an increasing function of the income $m$, for a fixed price system $q$, it is a monotonic transformation of the the minimum income function leading to the simple observation that the minimum income function is itself a “neoclassical, demand generating, indirect utility function (393).”

We shall have more to say on how a function that is to do away with a utility index itself becomes a utility index; for the moment, let us simply observe that the crucial insight of Hurwicz and Uzawa’s contribution, the one which McKenzie highlights, is that the demand functions $f(\cdot, \cdot)$, given what are referred to in the literature as Slutsky’s integrability conditions and some technical smoothness requirements, yield the minimum income function. This can be seen simply by differentiating equation (3.1) with respect to $q_i$ and using Roy’s identity, to obtain for each $i = 1, \cdots, n$,

$$\frac{\partial V}{\partial q_i} + (\frac{\partial V}{\partial M})(\frac{\partial M}{\partial q_i}) = 0 \implies (\frac{\partial M_f(p, m)(q)}{\partial q_i}) = f'(q, M_f(p, m)(q)).$$

If we now use the fact that $m = M_f(p, m)(p)$ as a boundary condition, we can solve the the partial differential equations (3.1) to obtain the function $M_f(p, m)(\cdot)$. Willig writes

49 See Part II of Morishima et al. (1973) and its references.
50 Willig’s text quotes these references in terms of bibliographic number rather than the date of publication.
One of the marvels of this theorem is that it contains the instructions ... for constructing an indirect utility function from a set of demand functions (393).

Now the analytical thrust of Willig’s paper is clear. He uses the hypotheses of his results, the same ones that have “escaped the notice of fifty years of demand theory,” to generate demand functions and thereby the indirect utility function as the minimum income function from whose form the conclusions of his results follow. And hence the importance of the minimum income function.

3.2 Integrability: A First Pass

In keeping with the title of his paper, Willig’s 1976 methodology falls squarely within what is referred to as the “integrability problem” originally associated with Slutsky and Samuelson: the question of conditions on under which a given system of demand functions, there exists a utility index such that the demands can be derived as the solution to the maximization of the index subject to a budget constraint. The question then arises as to how McKenzie’s minimum income function fits more generally into the integrability problem.

Hurwicz (1971) sets out clearly what question the minimum income function helps answer. One procedure, used by Samuelson (1950) is to make a transition from the demand-quantity relation to the demand-price relation and then integrate the demand-price relation. ...

But the inversion approach is somewhat roundabout and cannot be used when the demand-price relation is not single-valued. Yet the demand-quantity may be single valued, even though the demand-price relation is not...

A way out of this difficulty suggests itself in connection with the properties of the income compensation function (see McKenzie, 1957). This is so because income compensation functions generated by utility maximization satisfy differential equations involving the demand-quantity functions (i.e. the ‘direct demand functions’) as coefficients. Thus, it may be possible to integrate such equations, first obtaining income-compensation functions and then utility functions.

As the last subsection hinted, the minimum income function gives not just “a way out of” a difficulty in answering the question; it is itself an answer to the question. The representation $U_{p^*}(x)$ in equation (27) of Hurwicz-Uzawa (1971) is nothing other than the minimum income function, viewed as a function of the consumption plan for fixed prices $p^*$. The Hurwicz-Uzawa approach to integrability is now a commonplace in graduate texts in economic theory.

In the introduction to his 1980 book on Demand Functions and the Slutsky Matrix, Afriat invokes McKenzie (1956, 1957) twice: first, in connection with what he calls the necessity theory for the integrability problem; and second, as a further consideration

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51 For the 1886 contribution of Antonelli and the 1906 contributions of Pareto and Volterra, see Afriat (1980; p. 19-20). Also for references to Slutsky and Samuelson (1950).

52 In what follows the “demand-quantity relation” is the consumer’s demand function and the “demand-price relation” is the inverse demand.

53 See, for example, Mas-Colell, Whinston, and Green (1995, Section 3.H), and Kreps (2013, Section 11.5). Hurwicz and Uzawa (1971) are regularly cited in these texts, and rightly so. However, as mentioned in Footnote 8, the only text we know to cite McKenzie (1957) is Mas-colell, Whinston, and Green, (1995). In his excellent notes on the Uzawa-Hurwicz Integrability Theorem, Border (2004) points the reader to McKenzie (1957) in a footnote.
within the necessity theory, in connection with the problem of the Slutsky matrix being null. After outlining the sufficiency theory, Afriat writes

[T]he necessity is based on the approach of Lionel W. McKenzie (1956, 1957) shown in Chapter II. The framework in this is broader than Slutsky’s in that the direct utility function does not have to be differentiable, let alone continuously twice-differentiable, and correspondingly the demand function does not have to be invertible (19). The proofs of necessity of Slutsky conditions ... have the defect that the determinant of a bordered Hessian appears as a denominator without a proof that it is nonzero; an exception is the proof of Lionel McKenzie (1956, 1957) (22).

Afriat’s book is a consolidation of work stemming from 1953, and in particular, draws on his 1972 paper on “the case of the vanishing Slutsky’s matrix.” In his 1993 summary, Afriat refers to the minimum income function as a utility-cost function, and writes

Following McKenzie, differentiate a utility-cost function twice, and we have a negative semidefinite symmetric matrix of Slutsky coefficients. The important conditions are obtained simultaneously, in one short stroke. With Slutsky and others, after some pages of calculus, we find more is asked. Something extra, and in fact spurious, has entered unnoticed. ... [A]ll the coefficients may be zero—an impossibility in the original account. Here is evidence that the usually offered conditions are more than necessary. The conditions obtained by McKenzie, besides being necessary for the existence of a utility, appear good candidates for being sufficient also, as in fact they are.

A detailed treatment of these questions is available in Afriat (1980) whose Chapter II is titled “McKenzie’s Method.”

### 3.3 Money Metric and Observability

It is in 1974 that Samuelson renames the minimum income function the money-metric and charts out its use in the analysis of complementary commodities. The basic idea involves a change of perspective, one already at least implicit in Hurwicz-Uzawa: an emphasis on the commodity bundle $x$ in the minimum income function $M_x(p)$ instead of on $p$. Thus, if the price system $p$ is suppressed, once can simply write $M_x(p)$ as $M(x)$. Samuelson writes

If wished, one can speak of “utility,” $M(x)$, that is objectively measurable by money spendable at prices $p$. Mathematically $e(x)$ is given by $M(p;x)$, where the last function is defined behavioristically as $M(p;x) = \min_p py$ subject to $U(y) = U(x)$, where $U(x)$ is any indicator-numbering of the observable indifference contour (that for simplicity is assumed to exist and to have two or three continuous partial derivatives) (1262).

This is of course precisely the definition furnished by Yokoyama, and reproduced as equation (2.4) above, but with the difference that Yokoyama works more generally with a relation rather than with a utility index. Samuelson focusses on this difference.

It is more customary to write $M(p;x)$ as defined here in a form involving utility indicators, namely as $M(p|U(x))$ ... However, the present innovation has the merit of eschewing all non-observables, even in terms of representational symbols...Note that observing scientists have no need to rely on

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54 Afriat quotes McKenzie (1956) as McKenzie (1957b). For a recent flurry of interest in Afriat’s work, see Nishimura et al (2014), Reny (2014) and their references.

55 There is an interesting trail of the word *method* in this context: it occurs in the title of the *Cowles Foundation Discussion Paper*, (CFDP) McKenzie (1956) on which McKenzie (1957) is based. As pointed out in Footnotes 21 and 54, it is of interest that Afriat (1972) refers only to the CFDP, though with the incorrect date. And, as referred to in the introduction, McKenzie (2000) reverts to the terminology in his Chapter 1.
anyone’s psychological introspection to measure this \( M(p;x) \) function. Such measurements could be inferred from the objective demand functions.

This point that there is no need for a utility index receives constant emphasis in Samuelson’s article.

Indeed, by revealed preference data near \( (p,x) \) of the form \( (p^1,x^1), (p^2,x^2) \ldots \), one could hope to make all our qualitative and quantitative calculations. (If one wished for help from introspection, that introspection would not “involve” any mental “sensations,” but only estimates of how much one would buy at different prices and incomes!)

Samuelson’s essay is a \textit{tour de force} that combines literary and mathematical exposition with a masterful sensitivity to historical details. Under the subheading of \textit{New money-metric complementarity}, he defines the “basic minimized-expenditure functions that L. W. McKenzie (1957) and modern duality theory show to be so useful,” and notes that the resulting money-metric utility: (a) has an intuitive meaning, (b) involves no introspection about unobservables, (c) is symmetric for each of a pair of goods. However ... it does not give people the Gossen-like law of diminishing marginal utility of income (1263).” Referring to an indifference curve diagram of a conventional sort, but with reference to the money-metric utility function, Samuelson writes

\[ \text{We have resolved definitively, and in a non-introspective way that can be read off the vertical} \]
\[ \text{intercept, exactly what I mean by money-metric equally-spaced indifference contours (1262).} \]

The text of Fleurbaey-Maniquet (2011) is a testimony to the use of the money-metric in related fields, and we shall see in our subsequent consideration of the sixth measure of complementarity how Samuelson will

“\text{Weber-Fechnerize}” money-metric utility or, better still, use observable risk-aversion behavior to convert it into my von Neumann risk-utility metric (1263).

### 3.4 Complementarity

Samuelson (1974, p. 1255) begins his \textit{tour de force}\textsuperscript{56} on the concept of complementarity with the statement that the “last word has not yet been said on this ancient preoccupation of literary and mathematical economists.” He concludes his “verbal” exposition with the following summation.

Forty years is a long time to wait for a clarification of complementarity. Fortunately, and this tells us about the empirical importance of the SHAS\textsuperscript{57} digressions. Indeed there have been few econometric determinations of complementarity in all of this century, and fewer so refined that they could distinguish between the different esoteric definitions (1266).

In his discussion of six alternative meanings of the word \textit{complementarity}, the final two – presented as theoretical novelties of the paper – involve the minimum income function. Samuelson writes

\textsuperscript{56}Samuelson’s article is an antidote for the all too prevalent a malady that “history” and “theory” are substitutes, a point of view that, in its higher reaches substitutes “religion” and “science” for these two terms. It is an article in which Wittgenstein is constantly invoked, a total of eleven times, and one in which the main discussion, primarily literary, is followed by a mathematical section, and a “brief survey of the history of the subject.”

\textsuperscript{57}An acronym for Slutsky, Hicks, Allen and Schultze.
Now that we have a good solid utility metric to work with those who hanker for a solid and simple measure of complementarity can feel back in the promised land from which Pareto and others of us modern ordinalists drove them out (1262).

For his fifth definition, Samuelson proposes the following Axiom (or possibly a Convention).

Axiom: The indifference contours near one’s customary living standard, \( x = f(p, m) \), are to be given a cardinal numbering just equal to how much income, \( M \), would be needed at prevailing prices \( p \), to (most cheaply!) attain that new contour of living standard as in ordinary index-number theory (1262).

And he suggests as a definition

\[
\frac{\partial M(x_1, \ldots, x_n)}{\partial x_i \partial x_j} > 0 \text{ if and only if } i, j \text{ are complements.}
\]

Samuelson points to one remaining doubt he has about his fifth definition (beyond the lack of diminishing marginal utility): the drawback of “elevating complementarity above substitutability in the normal-goods case (1263).” Continuing to use Wittgenstein as his stalking horse, Samuelson notes that “his constituency may wonder whether are new definitions do capture the intuitive notions of the plain man (1264).”

Let us interrogate the fellow on this point [and ask:] Do you have some utility notion which, unlike that of he money-metric, does exhibit diminishing marginal utility of income in some intuitive, verifiable (i.e. non-private) sense? (1264)

Towards this end, Samuelson introduces a strictly-convex increasing function \( g_m(x) \) for “risk-averting me” that measures for each \( x \) lost, the “definite amount of gain you must give me at even odds to make me as well as as staying risklessly at \( m \). Moreover, as \( x \) losses rise in equal steps, needed \( g_m(x) \) gains rise at an increasing rate (1264).”

On the basis of your observable \( g_m(x) \) risk function, I (Wittgenstein, or F. Ramsey, or John von Neumann) can stretch your money-metric utility \( M_p(x) \) into a sixth concept of utility from which complementarity can be ideally computed (1264). So we are at our journey’s end, with an observable test for complementarity that has all the nice requirements ever felt to be needed, and which is subject to none of the numerous defects already alluded to (1265), namely

\[
\Phi(x) = b\phi(M_p(x)) + a, \quad b > 0, \text{ a arbitrary,}
\]

where \( \phi(\cdot) \) is “no longer an arbitrary \( f(\cdot) \) stretching, but has a definite \( \phi''(\cdot) \) curvature, which will be negative for me if I am risk-averse.

His sixth, and final definition, of complementarity reproduces his fifth with this \( \Phi(\cdot) \) function.

\[
\frac{\partial \Phi(x_1, \ldots, x_n)}{\partial x_i \partial x_j} > 0 \text{ if and only if } i, j \text{ are complements.}
\]

\footnote{We follow our convention in the quoting of texts.}
\footnote{Samuelson defines \( y \) to be equal to \( g_m(x) \) and uses that symbol in his next sentence. We reserve \( y \) for a general quantity situation; see Footnote 67 below.}
\footnote{What Wittgenstein’s constituency would make of Samuelson’s doubts about his fifth definition, and the content of his sixth, is an issue we hope to pursue in future work.}
3.5 Consumer’s Surplus

In his 1974 survey on *complementarity*, Samuelson addresses himself, with some “irony in view of [his] own long-standing lack of enthusiasm for consumer surplus integrals,” to all who have the “recurring desire to calculate consumers-surplus measures from observed demand curves.”

In the cases where these calculations are most valid, what is being computed is precisely my money-metric utility or its logarithm! To see all this, consider the case where one good, call it $x_1$, has literally constant marginal utility by some numbering of the indifference contours. Then that numbering agrees with money-metric utility; and the area under the independent $x_i$ good’s demand curve, $\int x_i dp_i$ (or in the case of independent demands, the line integral $\sum x_j dp_j$) will equal changes in $M_p(x)$, except for the dimensional factor $1/p_i$ (1264).

There is of course a veritable river of work on measures of consumers surplus, but staying within our assigned period of the 20th-century demand theory, and ignoring Dupuit and Marshall,61 we shall focus on two benchmarks subsequent to Samuelson’s 1974 article. Two different situations are involved: an initial price-income situation summarized by $p, m_p$ and a final situation $(q, m_q)$. The question is the economic well-being of the consumer in the two different situations – the ranking of the numbers $V(p, m_p)$ and $V(q, m_q)$. Willig (1976; p. 590), the first of our footholds, writes:

The compensating and equivalent variations can be most incisively studied and related to consumer’s surplus by means of the income compensation function (591).62 [A] way to effect the welfare test is to compare the income change $m_q - m_p$ to the smallest income adjustment needed to make the consumer indifferent to the change in prices from $p$ to $q$. If $m_q - m_p$ is larger, then welfare is greater in the new situation and inversely. Thus the compensating variation $C$ is an individual’s cost-benefit concept which makes price changes perfectly commensurable with changes in income. The equivalent variation $E$ is the income change which has the same welfare impact in the base situation as have the changes in prices from $p$ to $q$. The point is that

$$V(p, m_p) = V(q, m_q + C) \text{ with } V(p, m_p) \geq V(q, m_q) \iff m_q - m_p \geq C,$$

$$V(p, m_p - E) = V(q, m_p) \text{ with } V(q, m_p) \geq V(p, m_q) \iff m_p - E \geq m_q.$$

The trick is to see how the Marshall’s triangle under the uncompensated demand curve approximates $C$ and $E$. The “bridge to consumer’s surplus” is given by the following relationships:

$$C = M_f(p, m_p)(q) - M_f(p, m_q)(q) \text{ and } E = M_f(q, m_p)(q) - M_f(q, m_q)(p)$$

Now, using the formulae in Equation (3.1) above, along with the associated boundary condition, Willig (1976, Equations (12) and (13)) develops formulae for “compensating and equivalent variations as areas under demand curves, between the old and new price horizontals” but in terms of Hicksian compensated demands curves, the only “distinction between $C$ and $E$ [being] in the level of utility the compensation is designed to reach.”

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61 It goes without saying that their footprints are all over the subject and can hardly be ignored; what we mean to ignore is a consideration, exegesis if one prefers, of their texts.

62 Willig remarks in his Footnote 7, “This theoretical tool was introduced by Lionel McKenzie and definitively studied by Leonid Hurwicz and Hirofumi Uzawa.”
The issue then is how the area (integral) under the uncompensated (Marshallian) demand curve approximate these measures? Willig shows the answer to depend on the magnitude of the income elasticity of the commodity in question and on the proportion of income that is being spent on it, a fact that Hotelling (1938) was well-aware of; see Equations (1) and (2) in Willig (1976, p. 589).63

The second foothold we rely on is Hausman (1981). Commenting on Willig, he writes:

I show that for the case primarily considered by Willig of a single price change, which is also the situation in which consumer’s surplus is often used in applied work, no approximation is necessary. from an estimate of the demand curve, we can derive the measure of the exact consumer’s surplus, whether it is the compensating variation, equivalent variation, or some measure of utility change. No approximation is involved (662-663). The basic idea in deriving the exact measure of consumer’s surplus is to use the observed market demand curve to derive the unobserved compensated demand curve (663).64

It is the derivation of the indirect utility function and the expenditure function that allows Hausman to compute the compensated demands. But, since “estimates of the complete system of demand functions are not available, Hausman introduces and works with what he calls quasi indirect utility function and quasi expenditure functions. In an Appendix, Hausman computes the indirect utility function and the expenditure function which corresponds to a fully quadratic demand function.65

3.6 Index Numbers

Samuelson-Swamy (1974, p. 566) introduce a discussion of index numbers by listing Fisher’s 1922 properties that are desirable for an ideal index (in their first paragraph), and Frisch’s 1930 impossibility result for “well-behaved formulae that satisfy all these Fisher criteria (in their second paragraph.” In an aside on index numbers in his 1974 survey on complementarity, Samuelson had already written in the context of an economic theory of index numbers that:

A properly constructed economic index of quantity will give precisely the money-metric utility. Moreover, in the case where all people have the same homothetic tastes, adding their money-metric utilities (however illegitimate that is for any interpersonal welfare decisions) will definitely give us an exact measure of real social output that is free of the well-known index number difficulties (1264).

63Schlee (2013a) shows that consumers’ surplus approximates not only the compensated and equivalent variations, but four other measures, all in the context of general equilibrium model in which all relative prices can change: a local welfare measure proposed by Radner (1993), a Slutsky compensation, Debreu’s (1951) coefficient of resource utilization, and the Divisia quantity index (the last two measured in units of a numeraire good). The line integral that Samuelson introduces in the quotation that opens this subsection is fundamental. Schlee (2013b), proves a more powerful result—not involving approximations—in the partial equilibrium setting that Willig considers.

64Schlee (2007) points out some difficulties in the application of this result in the context of a random utility model.

65The computation is a veritable tour de force: a nonlinear differential equation, a Ricatti equation, is obtained as a consequence of Roy’s identity, and then transformed to by a change-of-variable technique to a “second order differential equation of the form studied by physicists.” By another change-of-variable technique, the original equation lands up as the “famous Schrodinger wave equation.” With some additional change-of-variable exercises, Hausman obtains formulae (A1 and A2) for the indirect utility function and the expenditure function – and all this in an AER article. All this is obvious input to the basic thesis in Mirowski (1989).
In 1974, Samuelson-Swamy still find the “best general references to be the 1936 surveys of Frisch, Leontief and Hans Staehl” and discern three different approaches to the subject. Of which the third they see as an “economic” one in which a price index is seen to equal the ratio of the (minimum) costs of a given level of living in two price situations and a quantity index which “measures for two presented quantity situations $x^0$ and $x^1$ the ratio of the minimum expenditure needed, in the face of a reference price situation $p,$ to buy their respective levels of well-being.”

The formal definitions are easily given. The fundamental point about an economic quantity index, which is too little stressed by writers, Leontief and Afriat being exceptions, is that it must be a cardinal indicator of ordinal utility. That is $q(x, y; p)$ must, for $p$ and $x$ fixed, be itself of the form $f\{U(y)\}$. Likewise $q(x, y; p)$ must, for $p$ and $y$ fixed, be itself of the form $-f\{U(x)\}$.

To present these beautiful theorems in this short review, applied mainstays of twentieth-century demand theory, will take us too far afield and divert us from our main purpose. The point to be emphasized is that the homothetic advance, which is what the articles contributes, is squarely based on the fact that derivative of the the minimum income function yields the compensated demand.

We conclude with the Samuelson-Swamy (1974, p. 592) warning, and exhortation.

Empirical experience is abundant that the Santa Claus hypothesis of homotheticity in tastes and in technical change is quite unrealistic. Therefore, we must not be bemused by the undoubted elegances and richness of the homothetic theory. Nor should we shoot the honest theorist who points out to us the unavoidable truth that non-homothetic cases of real life, one must not be able to make the naive measurements that untutored common sense always longs for; we must accept the sad facts of life, and be grateful for the complicated procedures economic theory devises.

### 3.7 Continuity, Duality and a Second Pass at Integrability

Weymark (1985) follows Samuelson’s terminology of a money metric and introduces his useful paper by sharply distinguishing “the expenditure function of McKenzie [1957] ... from what is now known as an expenditure function;” the latter depending on the utility level of the commodity bundle rather than the bundle itself, and thereby depending on proxying something introspectively grounded. We have already seen Samuelson’s use of the second differentials of the money-metric to formalize intuitive notions of complementarity, and so it stands to reason that we at least ask for conditions under which

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66 See Footnote 1 which dates Staehl to 1935, and mentions Samuelson’s *Foundations*, Pollak, Afriat and Richard Ruggles.

67 See Samuelson-Swamy (1974, p. 567). We follow our practice of making the notation conformable.

68 Note that in keeping with our convention, the initial price situation is represented by $p$ and the final by $q$; similarly $x$ and $y$ for the quantities, where $y$ is the final quantity situation. $p$ and $x$ are also used as generic symbols for a general situation.

69 In their proof of the equivalence of the invariance of the price and quantity indices, Samuelson-Swamy (1974, p. 570) use what they call “the classical formulation of Lionel McKenzie or the standard envelope theorem of maximization subject to a parameter.” The importance of the homotheticity assumption to unify and justify several welfare measures deserves emphasis. Gorman (1953) is a classic reference.

70 See a comprehensive overview of current thinking in Reddy-Plener (2006).

71 Though without the hyphen!
it is a utility function. Weymark attributes to Arrow-Hahn (1971) the observation that the conditions of Debreu’s 1959 theorem on the representation of preferences as utility functions are not sufficient for the problem at hand, but by considering consumption sets equal to the non-negative orthants of n-dimensional Euclidean space and continuous monotone preference orders, the result is easily obtained.\textsuperscript{72}

Weymark presents sufficient conditions for a money-metric to be well-defined and to be a utility function, and for them to be based only on continuous preferences and to be extendable to the entire consumption set. Weymark also presents results under which an indirect money-metric is also an indirect utility function. It is worth treating these concepts in some brief detail.

With a given nominal income $m$ consider a relation $\succ^*$ on the set $V$ of normalized prices $p/m$, and for a particular normalized price $w \in V$, the set $W(v) = \{w \in V : v \succ^* w\}$. If $\succ^*$ is representable by a function, it is called an indirect utility function. Then for a given quantity vector $x$, the indirect money-metric is defined to be

$$-Q_x(v) = \min_{w \in W(v)} xw.$$

Weymark (185, p. 230) writes

As is the case with money metrics, cost minimization is used to define and indirect money metric. However, now the quantity vector is fixed and the normalized price vector which is chosen in the optimization must be judged to be no better than the prespecified price vector $v$. Treating $x$ as a variable as well, $-Q$, (a function of prices and quantities) is the dual to McKenzie’s 1957 expenditure function. If instead of identifying an indirect indifference curve by $v$, it is indexed by a utility number, the function of quantity and utility corresponding to $-Q$ is known as the transformation function. This function is the symmetric dual to the expenditure function.\textsuperscript{73}

Already by the end of the seventies, Blackorby-Primont-Russell (1978) and Fuss-McFadden (1978) had been published, and Blackorby-Diewert (1979) could turn their attention to some “rather delicate questions” given that the ideas having to introduce their paper with the following statement:

Duality has become an increasingly important tool in many important branches of economics. This importance rests on two facts: (i) the specification of a functional form in prices entails (under suitable regularity conditions) the existence of of a functional form in quantities which contains the same information about preferences or technologies, and (ii) the differentiation of the functional form in prices yields expenditure minimizing (utility maximizing) derived demand (or supply functions).

To be sure, the 1974 Samuelson-Swamy paper on the special properties of index numbers in a homothetic world is steeped with duality, and the authors see the contribution of their work to four different aspects of neoclassical demand theory:

1) [The] duality theory of production and cost, elaborated by Shephard, Samuelson (1953), Hirofumi Uzawa, and Daniel McFadden; 2) the duality theory of direct and indirect utility, associated with Harold Hotelling, René Roy, Hendrik Houthakker (1960), Samuelson (1965) and Pollak; 3) the interrelations between the two dualities, as in Samuelson, and 4) certain new aspects of duality.

\textsuperscript{72}Weymark (1985) notes that the simplicity of the proof rivals that of the 1943 treatment of Wold. See our appendix where we present an 8-line proof. Also, in Footnote 1, he notes that the conditions furnished by Arrow-Hahn (1971, p. 106) are not sufficient.

\textsuperscript{73}We remind the reader of Wemark’s terminological distinction between the money-metric or the minimum income function, as it is being termed here, and the expenditure function. Weymark also notes that that the transformation function is also known as the distance or guise function. Also, see Deaton (1986).
3.8 Demand Theory without Transitivity

It follows from Samuelson (1938a) and Shafer (1974) that, if strict convexity replaces transitivity of preferences in McKenzie (1957), the weak axiom holds, and hence that the “generalized law of demand” holds for the Slutsky substitution terms $s_{ij}$ as well.\(^{74}\)

But, absent transitivity, the ordinary and compensated demands need not be equal, so in general $\frac{\partial h_i}{\partial p_j} \neq s_{ij}$. McKenzie (1957) needs transitivity precisely once (first line of of p. 187) to establish a result immediately implying that the (ordinary) demand evaluated at the minimum income function for a bundle $x$ solves the income minimization problem at $x$. It is essential for his derivation of the Slutsky equation, but neither invoked nor needed for any other result in the paper: that the income-minimization problem has a solution; that the minimum income function is concave in prices; that its derivative, when it exists, gives the solution $\mathbf{h} = (h_1, \ldots, h_n)$ to the income minimization problem (his equation (2)); and that these demands also satisfy a generalized law of demand (replace $s_{ij}$ with $\frac{\partial h_i}{\partial p_j}$ in the last footnote).

A theme of the next section is that a classic text says more than it seems to say.\(^{75}\)

Once the utility function is dispensed with, one is tempted to wield a more powerful instrument than Occam’s razor and learn what remains of consumer theory when either transitivity or completeness is dropped. The way was cleared for the elimination of transitivity by Georgescu-Roegen (1936) and Samuelson (1938)– whose consumers satisfy the weak, but not what later became known as the strong axiom—and for completeness by Aumann (1962).\(^{76}\) Indeed, in 1981 McKenzie extended his 1959 theorem on the existence of a competitive equilibrium to consumers without transitive preferences, and devoted one section of his 2002 text on classical general equilibrium theory to demand theory without transitivity as an end in itself, and not “just” as a part of equilibrium theory. In this section, he faces squarely the question of the existence of a demand function, and writes:

> The loss of transitivity has required a new and somewhat more difficult proof that the demand correspondence is well defined for strictly positive prices.\(^{77}\)

The proof uses the KKM lemma and thereby ushers fixed-point theory into partial equilibrium analysis; a subject on which we shall have more to say in the next subsection.\(^{78}\)

Fountain (1981) takes up the latent nontransitivity in McKenzie (1957), and asks whether consumer’s surplus calculated from a demand can be interpreted “either as a money measure of utility change or in terms of willingness to pay ideas.”\(^{79}\) Fountain

\(^{74}\)\(\sum_i \sum_j s_{ij} dp_i dp_j \leq 0\) for almost-all prices.

\(^{75}\)See Footnotes 5, 95 and 14, and the text they footnote.

\(^{76}\)In Samuelson (1950), the 1938 enthusiasm is lost for the notion that a consumer could satisfy the weak axiom, but still violate symmetry of the substitution terms, hence transitivity, implied by the strong axiom; also see Houthakker (1950).

\(^{77}\)McKenzie (2002, p. 7) is clear that in the absence of transitivity his Lemma 7 on showing that $z \in f(p, M_x(p))$ is indifferent to $x$ no longer holds.

\(^{78}\)This remarkable theorem was first proved by Sonnenschein (1971) using the KKM lemma, and later given another, elegant proof by Shafer (1974).

\(^{79}\)Nishimura et al. (2014) continues this investigation. In this connection, the following statement from Hicks (1956, Preface) is of interest. “It was Houthakker’s 1950 article, with its demonstration that transitivity is the ‘logical’ counterpart of integrability, which (from my point of view) supplied the missing link and tied the whole of the logical theory together.”
extends the Slutsky equation to the case of nontransitive preferences, with explicit appeal to the minimum income function. Decision theorists have of late renewed interest in incomplete, but not intransitive, preferences. The larger question remains of whether models which drop transitivity are of interest in economic theory. The profession has no settled view on whether transitivity or completeness is the less objectionable, but McKenzie (1981) is clear:

Whether the preference relation $\succ_i$ is complete seems to be a matter of definition, since $x$ incomparable with $y$ can be replaced harmlessly by $x$ indifferent to $y$ in the absence of transitivity.\(^80\)

Kim and Richter (1986) approvingly quote this passage, and go on to argue that “[W]e show that the recent emphasis on ‘incomplete’ preferences is somewhat misplaced. Indeed the same demands generated by ‘incomplete’ preferences, can often be generated by ‘complete’ preferences.” The point is that this issue is one regarding which McKenzie’s contribution to demand theory has failed to forge a consensus and a canon.\(^81\)

### 3.9 Individual to Aggregate Household Demand

It is a standard, well-established in the texts of Hicks, Samuelsen, Debreu and Arrow-Hahn, to build up general competitive analysis by an analysis of its individual producer and consumer constituents. McKenzie (2002) follows this convention, but with a twist that considers the theory of consumer behavior as pertaining to a household rather than to a single agent, and thereby draws on his 1955 treatment of interdependent consumer preferences. This has the important consequence that he can import the methods of general equilibrium and game theory into what is standard partial equilibrium analysis, being constituted precisely to articulate a general equilibrium. It is a masterful reaching-out that has not been fully appreciated, and deserves an extended treatment in its own right. Here we confine ourselves to brief remarks on Section 1.5 on market demand functions, and Appendix G on group demand functions. Incomplete and non-transitive preferences, and therefore Section 1.2, constitute a subtext to both.

The fact that one has to invoke fixed point theory to show the existence of an element in a compact set that is maximal for a binary relation that is incomplete and non-transitive is well-understood in mathematical economics, but McKenzie goes further. The final theorem of the chapter considers a finite number of households with interdependent preferences, much as in Cournot and Nash, and asserts the existence of a well-defined household demand correspondence as a function of strictly positive prices and the income of each household.\(^82\) In passing, McKenzie observes that the result reduces, in the case of a single household, to the individual demand of standard consumer theory.

Moreover, the general method of proof will apply to the existence of a competitive equilibrium. This comes from the fact that both of these results involve maximization of preferences, by one individual

\(^{80}\)Also see the first paragraph in McKenzie (2002, Section 1.2).

\(^{81}\)We are happy to acknowledge a referee’s suggestion to point out to the reader that models with nontransitive preferences are more controversial than the “other (by now completely canonized) previous applications.” Hicks (1956, p. 20) comments on the ‘completeness’ assumption: “This assumption is so unrealistic that it was bound to be a stumbling-block.”

\(^{82}\)This work can be seen as showing the existence of a mixed-strategy Nash equilibrium with a common action set but with payoffs generated by non-ordered preferences. It is interesting that Nash does not make it to McKenzie’s index.
Rather than the KKM lemma, the new techniques involve Michael’s selection theorem and the Kakutani theorem. But McKenzie’s 2002 theory of consumer demand draws not only on fixed-point theory, but also on the other staple of general competitive analysis, the separating hyperplane theorem.

In this change of emphasis from a single consumer to a household with many members, McKenzie is led naturally to Pareto efficient allocations of the household resources, and to the corresponding utility possibility frontiers. Under the assumption of a unique supporting vector \( \gamma \) depending on the prices and household income, the household demand function can be defined, and a surprising and seminal analogy exploited.

Just as in the case of the demand of the individual consumer, we may define \( M_x(p|\gamma) \) as the minimum income sufficient to allow the utility level \( U(x, \gamma) (= \gamma u) \) to be reached for a fixed \( \gamma \) when the prices are given by \( p \). Then, if \( \gamma \) is constant, then in complete analogy ... we can derive [the conclusion] of Theorem 2. However, to obtain the full effect of the price change, we must differentiate the demand function with respect to \( \gamma \), and \( \gamma \) with respect to \( p \).

Theorem 2 is none other than the Slutsky decomposition discussed in (6) above. But it is in Theorem 13, the last result of the chapter, based on the full effect of the price change, that the connection is made to the work of Browning-Chiappori, Diewert, Hildenbrand, Jerison and their followers. Theorem 7 based on another version of (6) above, one pertaining to interdependent consumers, had already connected to the work of Diewert (1982) and Gorman (1981). How the minimum income function threads through this rich and fascinating literature, a mixture of both theoretical and applied economics, must await a future occasion.

### 3.10 Compensated Demand in General Equilibrium

At the conclusion of his article introducing money-metric utility, Samuelson (1974; p. 1266) warns against its use in general competitive analysis. Whatever the merits of the money-metric utility concept developed here, a warning must be given against its misuse. Since money can be added across people, those obsessed by Pareto-optimality in welfare economics as against interpersonal equity may feel tempted to add money-metric utility across people and think there is an ethical warrant for maximizing the resulting sum. That would be an illogical perversion, and any such temptation should be resisted.

Samuelson’s warning relates of course to welfare economics; the issue that we want to consider in this section relates to the existence of competitive equilibrium. It is of interest that in Khan’s (1993) survey of Lionel McKenzie’s contributions to this question, his work on demand theory is not mentioned. Indeed, at first blush, the direction of influence seems to have gone only one way: from general equilibrium theory to that of demand. However, a more careful look reveals this not to be the case on two counts,

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83 McKenzie also points out that for the case where “maximization does not play a central role, a different approach must be taken,” and refers to his paper written in 1954, and to that of Debreu written in 1970.

84 Never the most scrupulous in terms of referencing, the bibliography of McKenzie (2002) does not list any paper of Gorman’s or Jerison’s.

85 The following warning is repeated in Samuelson’s discussion of index numbers; see the quote in the subsection above.
both developments of the early seventies: the analysis of *compensated equilibria* initiated in Arrow-Hahn (1971), and general equilibrium theory with indivisible commodities initiated by Henry (1970). Let us take each in turn.

Arrow-Hahn (1971) defined a *compensated equilibrium* by substituting the operation of utility maximization by expenditure minimization. Since the second welfare theorem asserts the existence of prices at which, under convexity, every Pareto-optimal allocation can be sustained as an expenditure minimizing bundle, it is a natural question ask whether one can show the existence of a compensated equilibrium using the techniques of Negishi (1960) and working on the Pareto frontier. It is this approach that is followed in Arrow-Hahn (1971) and Moore (1975). However, it was only in the late eighties that Honkapohja (1987) connected this existence problem to the minimum income function and attempted a direct proof using the fixed point arguments of Debreu and his own extensions of Weymark’s results on continuity of the money-metric. He writes:

I show how the continuity of the compensated demand correspondence can be used to construct a simple proof of the existence of a compensated equilibrium (548). My treatment avoids the use of utility functions altogether (549).

However, as Hammond (1993, Footnote 5) points out, this approach cannot assume free disposal unless trivialities are to be avoided. In any case, Honkapohja assumes convexity of the consumption set and of the preferences when he turns to the existence question.

General equilibrium theory is nothing, if not the analysis of the ensemble of individual demands, and it is somewhat bewildering that a pioneering contribution to individual demand theory, one freeing it from convexity and closedness, was not incorporated when aggregate demand was being considered.\(^{86}\) It is important that the point we are trying to make not be misunderstood: we are not referring to herding behavior or deeper issues concerning the sociology of knowledge, but simply that an important intellectual connection was missed. Thus Henry (1970) references only Debreu and McKenzie, and begins his paper as follows:


Dierker (1971) has only three references, Debreu and Henry being the only two to the economic literature. Henry (1972) proves the existence of approximate core allocations through the application of the theorems of Scarf and of Shapley-Folkman. Anderson-Khan-Rashid (1982, Remark) show the existence of approximate equilibrium in a setting with indivisible commodities. However, it is only in Broome (1972), Mas-Colell (1977), Khan-Yamazaki (1981), Khan-Rashid (1982), Hammond (1993) who exploit the existence of a perfectly divisible commodity, and here McKenzie’s Figure 1 is prescient. In this literature, it is expenditure minimization, and the weaker maximality notion associated

\(^{86}\)The relaxation of the closedness assumption is more of an afterthought: freeing the theory from the assumptions of convexity and smoothness of preferences are the deliberate and more influential considerations.
with it that serves as the opening; see Hammond (1993, p. 79) and also his discussion of the work of Khan-Yamazaki (1981).

A more comprehensive discussion of all this work is overdue, but this is not the place to undertake it. We now move on to a more relevant concern and conclude this section, not the subsection, with a summary statement that it elaborates for the reader all that has been built on McKenzie’s approach: the previous section, in contrast, focused on what McKenzie himself had done in the original papers.

4 Higher-level Theorizing: On the Framing of a Result

By 2002, it is not as an intruder but rather as an authority that Lionel McKenzie distinguishes three approaches to 20th-century demand theory: his direct method, calculus-based classical methods and the method of revealed preference.87 However, other than the quotation in the second paragraph of this essay, he does so with a studious silence as to any comparative ruling regarding them.88 In what sense then, one is led to ask, is the 1956-57 paper seminal, and the approach in it, primary? In this concluding section, we turn to this by using as essential background the the various strands that have so far been listed and explicated in Sections 2 and 3 above. We do so in a series of footholds and argumentative steps: (i) a renewed consideration of what we mean by the problematic of 20th-century demand theory, (ii) the difference the history of ideas and the history of economic thought, and within the latter, the distinction between a history of economic doctrine and a history of economic analysis, (iii) what we mean when we sight a text as saying more than it says, (iv) the difference between first and second-order theorizing, and finally, (v) McKenzie’s own self-understanding of his paper and of the subject the paper contributed to, both at the time of the writing and thereafter.89

Already the introduction is our first attempt to lay out what we see as the “problematic”, and to begin an explication of how the question regarding the nature of a classic, in the paragraph above, leads into a larger set of questions that go beyond McKenzie’s (1957) contribution per se to the general issue of how to understand the place of a theorem. When we ask how we are to place the minimum income function in the history of 20th-century demand theory, it is this crucially important distinction between a result and the context of that result that we are attempting to articulate. It is the difference between working inside a model and working outside it: comparing a particular model with its rivals, its others, so as to get a fix on the problematic of demand theory that they all are trying to model and find an expression for, and thereby find direction both

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87 We remind the reader that the classical starts from utility, the direct method from preferences, and revealed preference from demand; also see Footnote 1 below.

88 It may be pertinent to observe here that the 2002 text is a polishing of McKenzie’s Rochester notes, and so he is writing a bit more in the voice of a teacher, one confronting the three approaches and choosing not to take sides.

89 We present this listing, and the corresponding outline of this section, primarily as a reader’s guide for the micro-theorist with more narrowly-focused analytical interests. As an elaboration for such a reader, (i) continues the explication in Footnotes 10, 13 and 91 of the word “problematic” and draws further on Pippin’s review of Fried’s work; (ii) takes Condren and Kermode as its interlocutors, (iii) engages Stigler as its interlocutor, and given this background, (iv) is guided by Cavell’s (1969) work *Must we mean what we say?*
to the “right” model, as well as to the “right” context within which the model is to be read and evaluated. We use the previous review of McKenzie’s seventeen-paragraphed paper, its reading in Section 2 and its sixty-year old reception in Section 3, as the means towards an elucidation of this end: to see how the reading of papers considered canonical and classical, “might teach us to appreciate their “ontological status” and the historical fate of such self-understanding.”

The question as to the canonical and classical status of a text is hardly novel: it is a staple of the subject of intellectual history and the history of ideas.

... what is it that bestows classic status upon a political theory text? Why are some papers always read even if ... we are not often found reading them? The quintessential question, then, is what aids endurance?

Condren’s own free-ranging answers involve exploitation and use, emblem and authority, inherent and purposive ambiguity, all having to do with eristic debate and disputation. In particular, he states that the answers to his questions are to be “found not in an edifying catalogue of academic and literary virtues, but in the sleazy ambit of the most venerable of academic vices, ambiguity.” But our first, and somewhat sharply-expressed claim, is that this emphasis on ambiguity makes little sense for an economic text: when we substitute the word “economic” for “political” and ask instead

... what is it that bestows classic status upon an economic theory text?

the illegitimacy, if not the mischief, of the substitution becomes clear. The question does not migrate; there is a lack of complementarity between the two registers. It is far from clear how ambiguity is to be harnessed for, and to, a subject whose very raison d’etre and pride is to be free of it. We connect to Kermode’s classic text to fashion an answer.

We must cope with the paradox that the classic changes, yet retains its identity. It would not be read, and so would not be a classic, if we could not in some way believe it to be capable of saying more than its author meant; even if necessary, that to say more than he meant was what he meant to do. Some such way of thinking seems necessary to any view of the classic that will allow us to be its contemporaries.

We are back to the study of unintended consequences and Samuelson’s “serendipity of irrelevant issues,” though placed in an unconventional register. With this key, the two parts of this essay, Mackenzie’s text in Section 2 and its reception in Section 3, come together: the text must be capable of saying more than it says, that it brings

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90 To be sure, this is the key to unlocking what Samuelson is to call the “serendipity of irrelevant issues”; see Footnotes 5 and 95.
91 We shall return to these phrases in the sequel, but for the moment, in service of further explication of the use of the word “problematic,” see Pippin (2005; p. 578) referred to in Footnote 10, and the following quote from this text: “to tie painterly meaning not so much to working out problems in perception, mimesis, iconography, formal organization, and the like, but to the problem of genuine and false “modes of being of the artwork itself, all within a more general understanding of meaning-responsive beings at work in a social world in historical time.”
92 The first question concludes the first paragraph of Condren’s concluding chapter titled Toward an Explanation of Classic Status; for this and the other two questions, see Condren (1985, 253-254).
93 See Condren (1985, p. 260) where he refers to the answers as “means of transcendence.”
94 See F. Kermode (1983, p. 80), “The Classic: Literary Images of Permanence and Change,” Harvard University Press. He refers to this an “an imperialist formula,” and offers a vigorous defense against the “risk of it being thought a piece of trivial mysticism.” Our defense in the context of a classic economic text follows below.
95 As to the ironic thrust of McKenzie’s 1956-57 work, and for markers of Samuelson’s important phrase, see Footnotes 5, 14, 90 and the text they footnote.
together streams of work that flowed separately before, and in the couplings and joinings of inherited conceptual *apparati*, reveals colors and patterns that were not seen by the subject before.

But what does one mean when one says that McKenzie’s 1957 paper obviously said its author meant to say? and that to say more than he meant was what he meant to do? A theorem is not a poem or a sacred text with its concepts metaphorically slipping from one context to another; it is an if-then, or an if-and-only-if, statement saying only what is needed to be said in going from its hypotheses to its conclusions – to say that efficiently and parsimoniously, and to say no more. The point is that in seventeen paragraphs, and using as the instrument his construct of the minimum income function, McKenzie derived all of the testable implications of demand theory, and did so without a utility index, cardinal or ordinal.\(^{96}\) The point is that it only took 17 paragraphs, and most of those are preamble, a dramatically streamlined derivation. It is as we say in the Introduction a new proof of known results but one much shorter and more illuminating. In dispensing with convexity of preferences, and of the convexity of the domain on which these preferences were defined, in wresting the theory of individual demand from the language of “classical” optimization theory, and in placing it squarely in the substantial flow of convex analysis but with non-convexity as its overseer,\(^ {97}\) McKenzie’s 1957 paper said only that which needed to be said. And remaining at this level, there seems to be a logical leap from the claim that a paper can say more than it meant to say” to the claim made in the introduction that “that it can not be understood, much less extended and built upon, without keeping the problematic in view.” What is the need to go outside of microtheory to methodologists such as Condren, Cavell and Kermode?

The point, one that has been recurring throughout this exposition as its subtext, is that there are two registers involved: a purely technical register hinging on technical incrementation and clarifications and a larger context in which the theorem has to read and understood and which it helps in determining. It is a dialectical groping, a *tâtonnement* so to speak, in which, with time and distance, both the theorem and its context attain illumination. What is presented by its author as eliminating a utility index ends up by becoming a utility index (!), used by subsequent authors in answering questions of which the author himself may not have been aware. What he regards as the eliminating of inessential assumptions and a simple exercise of Occam’s razor becomes entry points for hitherto unexplored explorations, and refashionings of the subject. Such a groping and a mutual determination – not just the technical incrementation of a theorem and not just its framing without an understanding of the technical vernacular and its twists and turns – that jointly contributes to the advance of theory. The 1956-57 work is a classic both because it said much more than it meant to say, but also because it meant much more to others than it said – – the review in Section 3 simultaneously testifies to how its influence went beyond its author’s intention and how its reception over time at the

\(^{96}\)There is no presumption that others did not: Slutsky, Hicks and Allen also did this using classical calculus-based methods, and Houthakker (1950) had completed Samuelsons (1938) ambition using the method of revealed preference.

\(^{97}\)We refer to the dropping of the convexity assumption in the papers of McKenzie’s that we have been reading. Also see Yamazaki’s work in this context, particularly his 1981 paper.
hands of its professional others allowed one to see things in it that could not be seen or conceived at the time of its writing.

This distinction between the mathematics of economic theory, and the framing of the mathematics in the context of the larger economic problem that the mathematics is being used to articulate is important enough to draw a distinction between first and second-order theory. The latter is a contemplation of contemplation, “reflections of reflections as thought takes its own thinking as its subject.” This is an alternative phrasing of Kermode’s formula, one which relies on a result and its context as sources of meaning for both, and countering it by a point of view that is not unopposed to it. There is at least one first-order theorist of the first rank who has long been interested in what we are characterizing as theory of higher orders, and particularly second-order. Stigler (1967) introduces the question at hand with a dichotomy, one arm of which he labels as the “principle of scientific exegesis.”

The man’s central theoretical position is isolated and stated in a strong form capable of contradictions by the facts. The net scientific contribution, if any, of the man’s work is thus identified, amended if necessary, and rendered capable of evaluation and possible acceptance. The test of an interpretation is its consistency with the main analytical conclusions of the system of thought under construction. If the main conclusions of a man’s thought do not survive under one interpretation and do under another, the latter interpretation must be preferred. This rule of interpretation is designed to maximize the value of a theory to science.

Stigler opposes this to a second arm that he labels the “principle of personal exegesis.”

[It] seeks to determine what the man really believed, although this search has no direct relevance to scientific progress. One will then invoke a different criterion to choose between conflicting passages: that interpretation which fits best the style of the man’s thought becomes decisive.

So how is one to read McKenzie’s paper so that the reading avoids his beliefs; isolates his central theoretical position; makes it refutable and confronts it with facts; identifies his net, rather than gross, scientific contribution; assess its survival probabilities, and maximize its value to science?

Stigler is important for this essay precisely as a counterpoint to Kermode: by fixing the contemporaneous view of “science” as an evaluative criterion, he fixes the context, and in this closure of higher-order thinking, valorizes the exclusions and what is taken as the tacit and the essential. Thus, the entire tenor of our argumentative thrust stands at an angle to Stigler’s findings. To be sure, we are not writing the history of economic doctrine, of 18th-century arguments for free trade, or of the 17th-century views of the poor as a productive resource, or even investigating the stance of a theorist towards, say, behavioral economics. It is here that we connect again to “ambiguity” and to the difference between an economic and a political text. The history of economic analysis...

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98 See the preface of Hampshire’s *Spinoza and Spinozism*, Oxford University Press. As such, there is no bar to the closure of orders, as one is led from second to third- to higher-order theory. The authors hope to follow these leads elsewhere.

99 This is further underscored in Footnote 105 below.

100 A detailed elaboration of this implicit, but rather forceful, contestation of Stigler is beyond the scope of this essay and will be elaborated elsewhere. Also see our implicit criticism of Condren above in the context of the reading of economic texts.

101 See, for example, Chapter 6 in Appleby (1978) titled “The Poor as a Productive Resource.”

102 The reader has a rich choice of first-order theorists to choose from in Caplin-Schotter (2008), for example.
and its context of high theory, in particular, is most favorable to Stigler’s position. The point is that for such a history, instead of figures of speech going out of control, there are figures of mathematics that do the controlling and the policing. But the besting of one temptation leads to another: does mathematics take the place of Stigler’s science and thereby control the history of economic analysis to such an extent that the very autonomy of the economic is irretrievably lost? Does the subject move with such a momentum that it is simply a matter of quickly, and quietly, getting into line, of spotting and picking the dollar bill on the ground before anyone else, a contribution more a testimony to the maturity of the investigation and to the sociological savvy of the investigator than to her creativity? The claim is that in this most favorable context of high theory – demand theory without a utility index, Stigler’s imperatives – what a theorist, or an evaluator of theorists, ought or ought not to do – are necessarily to be bracketed. The criteria for the writing of the history of economic doctrine do not suffice; they need supplementation to answer the question of what constitutes a classic in the history of economic analysis. It is a regress of reception that time accentuates; the result is used to determine the context, and the context then used to give meaning to the result, each provisional conclusion a foothold for the next. We need our invocation of Kermode.

But to determine whether a text says more than it actually says, of gauging the significance of a text in the light of the literature to which it contributes, is not a simple matter. Time and distance are surely a prerequisite for success, but the real danger is to draw on criteria that are not independent of all that they are a criteria for, that the judge is not himself fashioned by that which he is to judge, and on which he is to rule. How can a text be evaluated by the criteria that it itself forged? to evaluate prices by the criteria of the market-maker who makes them. It is here that the position that we lay out in this essay is orthogonal to that of Stigler. In the words of Samuelson (1974, p. 1264).

For how could it be otherwise? If you are measuring utility by money, it must remain constant with respect to money; a yardstick cannot change in terms of itself.

The point is prior to what and how the theorist should theorize: it is rather how she reads the subject to which, and to whose theory, she wants to contribute. How is she to determine the high points of the subject, the classical texts on which the subject rests and defines itself? The claim of this essay is simply that such a reading proceeds on two levels each of which are free-floating and rely on the other: the result and its context. It is this that leads us to emphasize higher and second-order theory: theorizing about the process of theorizing as being indispensable to economic theorizing. The history of thought being indispensable to the technical engine of the thought itself. This perhaps places one in the paradoxical position of gauging the importance of a paper not

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103 This point may do with further elaboration. What we have in mind is that mathematics, in irrevocably fixing the meaning of terms – they are to mean one thing and not any other – and the technical register which is the source of those terms, there is a shutting-off of a metaphorical slippage that characterizes a poem or a literary text. Thus, a model formulated within the confines of measure theory is under pressure to remain within its confines.

104 To underscore the point more directly, a canonical model begins to substitute itself for the economic phenomenon it models: the economy being viewed synonymously, or naturally, as the Arrow-Debreu model.

105 This is also why Stigler is rather important to this essay.
by the number of times that it is cited, or by whom it is cited, but by whether it is cited at all. Like Gauss-Markov for ordinary least squares, Kuhn-Tucker for first-order conditions, Dantzig for complementary slackness, the influence of the contribution could be so pervasive that a citation would be pedantry, a name need only be treated as an adjective.\footnote{This irony would not likely have escaped Lionel McKenzie; also see Khan (1993b) and Footnote 5.} In seeking a solution to a well-defined and limited problem, a classical paper is one in which the solution spills over beyond itself, the author is seen to transcend over time his own original intention, beyond the problem itself to a determination of what one means by a solution, on how things ought to be solved. It gently leads economic theorists towards re-learning the meaning of the phrase \textit{Occam's razor} to get criterial alternatives back on track.

And so we conclude this essay with the very words of McKenzie that began it.

I may be forgiven for intruding here a contribution to demand theory which was not itself a major advance but which led to some developments in the hands of Leo Hurwicz and his collaborators.

An alternative proof of results that were known already, not in itself a major advance, but an attempt to work out a particular model led to beyond the model and changed the way the subject was to be modeled: to question prevailing views regarding inclusions and exclusions, essentials from the inessentials, tacit from the explicit. McKenzie’s paper said a lot more than it meant to say, and became a point of reorientation and redefinition of the subject to which it sought merely to contribute.

5 Appendix: The Minimum Income Function as a Utility Index

Here we confirm that, for fixed prices, McKenzie’s Minimum Income Function represents the consumer’s preferences (on the subset of the consumption set that satisfies a version of McKenzie’s Locally Cheaper Points assumption). The argument is elementary and—the important point for us—requires little beyond what is already in McKenzie (1957). There are $n$ goods. Following McKenzie (1957), we take the consumption set $S$ to be any nonempty, closed subset of $\mathbb{R}^n$ that is bounded below. In particular, $S$ need not be convex. The consumer’s preference relation, $\succsim$, is a complete and transitive binary relation on $S$ that is continuous (for every $x \in S$, the sets $\{y \in S | x \succsim y\}$ and $\{y \in S | y \succsim x\}$ are closed).

As in the text, for $x \in S$, $C_x = \{x' \in S | x' \succsim x\}$ is the at-least-as-good-as set at $x$. To emphasize the dependence on the consumption point $x$, we write McKenzie’s minimum income function as

\[ M(p, x) = \min_{x' \in C_x} p \cdot x', \text{ where } p \in \mathbb{R}_{++}^L \text{ and } x \in S. \]

The existence of a solution follows from continuity and completeness. From now on fix $p > 0$. McKenzie (1957) imposes two more assumptions on $\succsim$.

\textbf{Assumption 1} (Local Nonsatiation). \textit{For every $x \in S$ and any open neighborhood $N$ of $x$ in $\mathbb{R}^n$ there is an $\hat{x} \in N \cap S$ with $\hat{x} \succ x$.}

By Local Nonsatiation, $p \cdot x' = m$ for any $x' \in f(p, m)$, the (ordinary) demand correspondence. McKenzie (1957) also imposes a Cheaper Point condition at a single point $x \in f(p, m)$. We impose a variant of it.

\textbf{Assumption 2} (Local Cheaper Point). \textit{Given $x \in S$, for every $x' \in f(p, M(p, x))$, and open neighborhood $N$ of $x'$ in $\mathbb{R}^n$, there is a point $\tilde{x} \in S \cap N$ with $p \cdot \tilde{x} < p \cdot x'$.}

Let $\tilde{S}$ be the set of points $x \in S$ for which the Locally Cheaper Point assumption holds; that is, $\tilde{S}$ is the set of points $x \in S$ such that, at any point demanded at $(p, M(p, x))$, there are arbitrarily close cheaper points.
Theorem 1. Suppose $\succeq$ is complete, transitive, and continuous; and that the Local Nonsatiation assumption holds. Then $M(p, \cdot)$ is a continuous representation of $\succeq$ on $\tilde{S}$.

Proof: Let $x, y$ be points in $\tilde{S}$. Suppose that $x \succeq y$. Since $\succeq$ is transitive, $C_x \subset C_y$, so $M(p, x) \geq M(p, y)$. Suppose now that $M(p, x) \geq M(p, y)$. McKenzie (1957: 186-7) uses completeness, transitivity (these two implicitly), continuity, local nonsatiation, and the local cheaper point assumption to confirm that $x \sim x'$ for any $x' \in f(p, M(p, x))$: By Local Nonsatition, $p \cdot x' = M(p, x)$; at any point $x''$ that solves (7), $p \cdot x'' = M(p, x)$, so $x' \succeq x''$; also, $x'' \succeq x$; by transitivity, $x' \succeq x$; by the locally cheaper point assumption at $x$ there is a sequence $x^n$ with limit $x'$ and $p \cdot x^n < M(p, x)$; since $\succeq$ is complete, $x \succ x^n$ for every $n$; by continuity, $x \succeq x'$. Now let $\bar{x} \in f(p, M(p, x))$ and $\bar{y} \in f(p, M(p, y))$, so $\bar{x} \sim x$ and $\bar{y} \sim y$. Since $M(p, x) \geq M(p, y)$, $\bar{x} \succeq \bar{y}$. Apply transitivity twice to conclude $x \succeq y$. We omit the proof that $M(p, \cdot)$ is continuous.

The theorem asserts that $M(p, \cdot)$ represents $\succeq$ on the set $\tilde{S}$ that satisfies the Local Cheaper Points Assumption. The restriction to that set ensures that the constraint in (7) that $x' \in C_x$ is not slack: if it were, a strictly preferred bundle might not require higher income. Of course, if $S = \mathbb{R}^n_+$ and $\succeq$ is monotone, then $S = \tilde{S}$.
Primary Sources: 1961 and Before


Secondary Sources


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We use Houthakker’s 1961 survey to draw the chronological boundary between primary and secondary sources.


